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Double posets and the antipode of QSym

Darij Grinberg*

University of Minnesota, School of Mathematics, 206 Church St. SE, Minneapolis, MN 55455, USA

Abstract. A quasisymmetric function is assigned to every double poset (that is, every finite set endowed with two partial orders) and any weight function on its ground set. This generalizes well-known objects such as monomial and fundamental quasisymmetric functions, (skew) Schur functions, dual immaculate functions, and quasisymmetric (P, ω)-partition enumerators. We prove a formula for the antipode of this function that holds under certain conditions (which are satisfied when the second order of the double poset is total, but also in some other cases); this restates (in a way that to us seems more natural) a result by Malvenuto and Reutenauer, but our proof is new and self-contained. We generalize it further to an even more comprehensive setting, where a group acts on the double poset by automorphisms.

Keywords: antipodes, double posets, Hopf algebras, posets, P-partitions, quasisymmetric functions

1 Introduction

Double posets and **E**-partitions (for **E** a double poset) have been introduced by Claudia Malvenuto and Christophe Reutenauer [14] in their definition of a "Hopf algebra of double posets". We shall employ these same notions to study a formula for the antipode in the Hopf algebra QSym of quasisymmetric functions due to (the same) Malvenuto and Reutenauer [13, Theorem 3.1]. We shall restate this formula in a more natural form, outline a new (and self-contained) proof, and extend it further to a setting in which a group acts on the double poset. This latter extension, and with it the whole work, owes its inspiration to Katharina Jochemko's [10]. This extended abstract surveys the results in [8] and sketches the main ideas of the proofs. For details, we refer to [8].

2 Notations

We set $\mathbb{N} = \{0, 1, 2, \ldots\}$. A *composition* means a finite sequence of positive integers. We let Comp be the set of all compositions. For any composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, set $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$.

*darijgrinberg@gmail.com

Fix a commutative ring $\mathbf{k}$. We consider the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series in infinitely many (commuting) indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. A *monomial* shall always mean a monomial (without coefficients) in the variables $x_1, x_2, x_3, \ldots$. The algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ comes with a topology that provides meaning to certain infinite sums; see [8, §2] for details. A power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *bounded-degree* if there exists a $d \in \mathbb{N}$ such that no monomial of degree $> d$ appears in f.

If two monomials $\mathfrak{m}$ and $\mathfrak{n}$ have the forms $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_\ell}^{a_\ell}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_\ell}^{a_\ell}$ for two strictly increasing sequences $(i_1 < i_2 < \cdots < i_\ell)$ and $(j_1 < j_2 < \cdots < j_\ell)$ of positive integers and one (common) sequence $(a_1, a_2, \ldots, a_\ell)$ of positive integers, then $\mathfrak{m}$ and $\mathfrak{n}$ are said to be *pack-equivalent*.[2] A power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *quasisymmetric* if every two pack-equivalent monomials have equal coefficients in front of them in f. The set of quasisymmetric bounded-degree power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, and is known as the $\mathbf{k}$*-algebra of quasisymmetric functions over* $\mathbf{k}$. It is denoted by QSym. It is clear that the symmetric bounded-degree power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (commonly known in combinatorics as the *symmetric functions*) form a $\mathbf{k}$-subalgebra of QSym. The quasisymmetric functions have a rich theory which is related to, and often sheds new light on, the classical theory of symmetric functions; this theory goes back to Gessel [5] and Malvenuto and Reutenauer [12], and expositions can be found in [17, §§ 7.19, 7.23] and [9, §§5-6] and other sources.

For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathrm{Comp}$, we set

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{\substack{\mathfrak{m} \text{ is a monomial} \\ \text{pack-equivalent to } x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}}} \mathfrak{m}$$

(where the i_k in the first sum are positive integers). Then, $(M_\alpha)_{\alpha \in \mathrm{Comp}}$ is known to be a basis of the $\mathbf{k}$-module QSym; it is known as the *monomial basis* of QSym.

The $\mathbf{k}$-algebra QSym can be endowed with a structure of a $\mathbf{k}$-coalgebra which, combined with its $\mathbf{k}$-algebra structure, turns it into a Hopf algebra. We refer to the literature both for the theory of coalgebras and Hopf algebras (see [16], [9, §1], [15, §1-§2], etc.) and for a deeper study of the Hopf algebra QSym (see, e.g., [9, §5]); we shall need but the very basics of this structure, and so it is only them that we introduce.

We define a $\mathbf{k}$-linear map $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$ (here and in the following, all tensor products are over $\mathbf{k}$ by default) by requiring that

$$\Delta\left(M_{(\alpha_1, \alpha_2, \ldots, \alpha_\ell)}\right) = \sum_{k=0}^{\ell} M_{(\alpha_1, \alpha_2, \ldots, \alpha_k)} \otimes M_{(\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_\ell)} \qquad \text{for every } (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathrm{Comp}.$$

(By linearity, this defines Δ on all of QSym, since $(M_\alpha)_{\alpha \in \mathrm{Comp}}$ is a basis of the $\mathbf{k}$-module QSym.) We further define a $\mathbf{k}$-linear map $\varepsilon : \mathrm{QSym} \to \mathbf{k}$ by requiring that

$$\varepsilon\left(M_{(\alpha_1, \alpha_2, \ldots, \alpha_\ell)}\right) = \delta_{\ell,0} \qquad \text{for every } (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathrm{Comp}.$$

[2] For instance, $x_2^2 x_3 x_4^2$ is pack-equivalent to $x_1^2 x_4 x_8^2$ but not to $x_2 x_3^2 x_4^2$.

(Equivalently, ε sends every power series $f \in \mathrm{QSym}$ to the result $f(0,0,0,\ldots)$ of substituting zeroes for the variables $x_1, x_2, x_3, \ldots$ in f. The map Δ can also be described in such terms, but with greater difficulty [9, (5.3)].) It is well-known that these maps Δ and ε satisfy the equalities

$$\left(\Delta \otimes \mathrm{id}_{\mathrm{QSym}}\right) \circ \Delta = \left(\mathrm{id}_{\mathrm{QSym}} \otimes \Delta\right) \circ \Delta, \quad \left(\varepsilon \otimes \mathrm{id}_{\mathrm{QSym}}\right) \circ \Delta = \iota_1, \quad \left(\mathrm{id}_{\mathrm{QSym}} \otimes \varepsilon\right) \circ \Delta = \iota_2$$

(where $\iota_1 : \mathrm{QSym} \to \mathbf{k} \otimes \mathrm{QSym}$ and $\iota_2 : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathbf{k}$ are the canonical isomorphisms), and so $(\mathrm{QSym}, \Delta, \varepsilon)$ is what is commonly called a $\mathbf{k}$-*coalgebra*. Furthermore, Δ and ε are $\mathbf{k}$-algebra homomorphisms, which is what makes this $\mathbf{k}$-coalgebra QSym into a $\mathbf{k}$-*bialgebra*. Finally, let $m : \mathrm{QSym} \otimes \mathrm{QSym} \to \mathrm{QSym}$ be the $\mathbf{k}$-linear map sending every pure tensor $a \otimes b$ to ab, and let $u : \mathbf{k} \to \mathrm{QSym}$ be the $\mathbf{k}$-linear map sending $1 \in \mathbf{k}$ to $1 \in \mathrm{QSym}$. Then, there exists a unique $\mathbf{k}$-linear map $S : \mathrm{QSym} \to \mathrm{QSym}$ satisfying

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta. \tag{2.1}$$

This map S is known as the *antipode* of QSym. It is known to be an involution and an algebra automorphism of QSym, and its action on the various quasisymmetric functions defined combinatorially is the main topic of this note. The existence of the antipode S makes QSym into a *Hopf algebra*.

3 Double posets

Next, we shall introduce the notion of a double poset, following [14].

Definition 1. (a) We shall encode posets as pairs $(P, <)$, where P is a set and $<$ is a strict partial order (i.e., an irreflexive, transitive and antisymmetric binary relation) on the set P; this relation $<$ will be regarded as the smaller relation of the poset.

(b) If $<$ is a strict partial order on a set P, and if $a \in P$ and $b \in P$, then we say that a and b are $<$-*comparable* if either $a < b$ or $a = b$ or $b < a$. A strict partial order $<$ on a set P is said to be a *total order* if and only if every two elements of P are $<$-comparable.

(c) If $<$ is a strict partial order on a set P, and if $a \in P$ and $b \in P$, then we say that a is $<$-*covered by* b if we have $a < b$ and there exists no $c \in P$ satisfying $a < c < b$.

(d) A *double poset* is defined as a triple $(E, <_1, <_2)$ where E is a finite set and $<_1$ and $<_2$ are two strict partial orders on E.

(e) A double poset $(E, <_1, <_2)$ is said to be *special* if the relation $<_2$ is a total order.

(f) A double poset $(E, <_1, <_2)$ is said to be *tertispecial* if it satisfies the following condition: If a and b are two elements of E such that a is $<_1$-covered by b, then a and b are $<_2$-comparable.[3]

(g) If $<$ is a binary relation on a set P, then the *opposite relation* of $<$ is defined to be the binary relation $>$ on the set P defined by the equivalence $(e > f) \iff (f < e)$. Notice that if $<$ is a strict partial order, then so is the opposite relation $>$ of $<$.

Definition 2. If $\mathbf{E} = (E, <_1, <_2)$ is a double poset, then an $\mathbf{E}$-*partition* shall mean a map $\phi : E \to \{1, 2, 3, \ldots\}$ such that:

- every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\phi(e) \leq \phi(f)$;
- every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\phi(e) < \phi(f)$.

Example 3. The notion of an $\mathbf{E}$-partition (which was inspired by the earlier notions of P-partitions and (P, ω)-partitions as studied by Gessel and Stanley[4]) generalizes various well-known combinatorial concepts. For example:

- If $<_2$ is the same order as $<_1$ (or any extension of this order), then $\mathbf{E}$-partitions are weakly increasing maps from the poset $(E, <_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$.
- If $<_2$ is the opposite relation of $<_1$ (or any extension of this opposite relation), then $\mathbf{E}$-partitions are strictly increasing maps from the poset $(E, <_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$.

For a more interesting example, let $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be two partitions such that $\mu \subseteq \lambda$. (See [9, §2] for the notations we are using here.) The skew Young diagram $Y(\lambda/\mu)$ is then defined as the set of all $(i, j) \in \{1, 2, 3, \ldots\}^2$ satisfying $\mu_i < j \leq \lambda_i$. On this set $Y(\lambda/\mu)$, we define two strict partial orders $<_1$ and $<_2$ by

$$(i, j) <_1 (i', j') \iff (i \leq i' \text{ and } j \leq j' \text{ and } (i, j) \neq (i', j')) \qquad \text{and}$$
$$(i, j) <_2 (i', j') \iff (i \geq i' \text{ and } j \leq j' \text{ and } (i, j) \neq (i', j')) .$$

The resulting double poset $\mathbf{Y}(\lambda/\mu) = (Y(\lambda/\mu), <_1, <_2)$ has the property that the $\mathbf{Y}(\lambda/\mu)$-partitions are precisely the semistandard tableaux of shape λ/μ. (Again, see [9, §2] for the meaning of these words.)

[3] The notions of a double poset and of a special double poset come from [14]. See [4] for more about the latter. The notion of a "tertispecial double poset" (in hindsight, "locally special" would be better, but other authors have already adopted this one) appears to be new and arguably sounds artificial, but is the most suitable setting for the results below (and appears in nature, beyond the particular case of special double posets – see Example 3).

[4] See [6] for the history of these notions, and [5] and [17, §7.19] for some of their theory. Mind that these sources use different and sometimes incompatible notations – e.g., the P-partitions of [6] differ from those of [5] by a sign reversal.

This double poset $\mathbf{Y}(\lambda/\mu)$ is not special (in general), but it is tertispecial. Some authors prefer to use a special double poset instead, which is defined as follows: We define a total order $<_h$ on $Y(\lambda/\mu)$ by

$$(i,j) <_h (i',j') \iff (i > i' \text{ or } (i = i' \text{ and } j < j')).$$

Then, $\mathbf{Y}_h(\lambda/\mu) = (Y(\lambda/\mu), <_1, <_h)$ is a special double poset, and the $\mathbf{Y}_h(\lambda/\mu)$-partitions are precisely the semistandard tableaux of shape λ/μ.

We now assign a certain formal power series to every double poset:

Definition 4. If $\mathbf{E} = (E, <_1, <_2)$ is a double poset, and $w : E \to \{1,2,3,\ldots\}$ is a map, then we define a power series $\Gamma(\mathbf{E}, w) \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$\Gamma(\mathbf{E}, w) = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_{\pi,w}, \qquad \text{where } \mathbf{x}_{\pi,w} = \prod_{e\in E} x_{\pi(e)}^{w(e)}.$$

The following fact is easy to see:

Proposition 5. *Let* $\mathbf{E} = (E, <_1, <_2)$ *be a double poset, and* $w : E \to \{1,2,3,\ldots\}$ *be a map. Then,* $\Gamma(\mathbf{E}, w) \in \mathrm{QSym}$.

Example 6. Various well-known quasisymmetric functions can be written as $\Gamma(\mathbf{E}, w)$:

(a) If $\mathbf{E} = (E, <_1, <_2)$ is a double poset, and $w : E \to \{1,2,3,\ldots\}$ is the constant function sending everything to 1, then $\Gamma(\mathbf{E}, w) = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_\pi$, where $\mathbf{x}_\pi = \prod_{e\in E} x_{\pi(e)}$. We shall denote this power series $\Gamma(\mathbf{E}, w)$ by $\Gamma(\mathbf{E})$; it is exactly what has been called $\Gamma(\mathbf{E})$ in [14, §2.2]. All results proven below for $\Gamma(\mathbf{E}, w)$ can be applied to $\Gamma(\mathbf{E})$, yielding simpler (but less general) statements.

(b) If $E = \{1, 2, \ldots, \ell\}$ for some $\ell \in \mathbb{N}$, if $<_1$ is the usual total order inherited from $\mathbb{Z}$, and if $<_2$ is the opposite relation of $<_1$, then the special double poset $\mathbf{E} = (E, <_1, <_2)$ satisfies $\Gamma(\mathbf{E}, w) = M_\alpha$, where α is the composition $(w(1), w(2), \ldots, w(\ell))$.

(c) Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition, and set $n = |\alpha|$. Let $D(\alpha)$ be the set $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\}$. Let E be the set $\{1, 2, \ldots, n\}$, and let $<_1$ be the total order inherited on E from $\mathbb{Z}$. Let $<_2$ be some partial order on E with the property that $(i + 1 <_2 i$ for every $i \in D(\alpha))$ and $(i <_2 i + 1$ for every $i \in \{1, 2, \ldots, n-1\} \setminus D(\alpha))$. (There are several choices for such an order; in particular, we can find one which is a total order.) Then,

$$\Gamma((E, <_1, <_2)) = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{\beta \text{ is a composition;} \\ |\beta| = n;\ D(\beta) \supseteq D(\alpha)}} M_\beta.$$

This power series is known as the α-th *fundamental quasisymmetric function*, usually called F_α (in [5], [12, §2], [2, §2.4] and [7, §2]) or L_α (in [17, §7.19] or [9, Def. 5.15]).

(d) Let $\mathbf{E}$ be one of the two double posets $\mathbf{Y}(\lambda/\mu)$ and $\mathbf{Y}_h(\lambda/\mu)$ defined as in Example 3 for two partitions μ and λ. Then, $\Gamma(\mathbf{E})$ is the skew Schur function $s_{\lambda/\mu}$.

(e) Similarly, *dual immaculate functions* as defined in [2, §3.7] can be realized as $\Gamma(\mathbf{E})$ for appropriate $\mathbf{E}$ (see [7, Proposition 4.4]), which helped the author prove one of their properties [7]. (The $\mathbf{E}$-partitions here are the so-called *immaculate tableaux*.)

(f) When the relation $<_2$ of a double poset $\mathbf{E} = (E, <_1, <_2)$ is a total order (i.e., when the double poset $\mathbf{E}$ is special), the $\mathbf{E}$-partitions are precisely the reverse (P, ω)-partitions (for $P = (E, <_1)$ and ω being a labelling of P dictated by $<_2$) in the terminology of [17, §7.19], and the power series $\Gamma(\mathbf{E})$ is the $K_{P,\omega}$ of [17, §7.19].

4 The antipode theorem

We are now ready for the main results. We first state a theorem and a corollary which are not new, but will be reproven in a novel and self-contained way.

Theorem 7. *Let $(E, <_1, <_2)$ be a tertispecial double poset. Let $w : E \to \{1, 2, 3, \ldots\}$. Then, $S(\Gamma((E, <_1, <_2), w)) = (-1)^{|E|} \Gamma((E, >_1, <_2), w)$, where $>_1$ denotes the opposite relation of $<_1$.*

Corollary 8. *Let $(E, <_1, <_2)$ be a tertispecial double poset. Then, $S(\Gamma((E, <_1, <_2))) = (-1)^{|E|} \Gamma((E, >_1, <_2))$, where $>_1$ denotes the opposite relation of $<_1$.*

We shall give examples for consequences of these facts shortly (Example 11), but let us first explain where they have already appeared. Corollary 8 is equivalent to [9, Corollary 5.27] (a result found by Malvenuto and Reutenauer [13, Lemma 3.2]). Theorem 7 is equivalent to Malvenuto's and Reutenauer's [13, Theorem 3.1].[5] We believe that our versions of these facts are more natural and simpler than the ones appearing in existing literature (and if not, at least our proofs are).

To these known results, we add another, which seems to be unknown so far (probably because it is far harder to state in the terminologies of (P, ω)-partitions or equality-and-inequality conditions appearing in literature). First, we need to introduce some notation:

Definition 9. Let G be a group, and let E be a G-set.

(a) Let $<$ be a strict partial order on E. We say that G *preserves the relation* $<$ if every $g \in G$, $a \in E$ and $b \in E$ satisfying $a < b$ satisfy $ga < gb$.

[5] These equivalences are not totally obvious. See [8, §4] for a few more details on them.

(b) Let $w : E \to \{1, 2, 3, \ldots\}$. We say that G *preserves* w if every $g \in G$ and $e \in E$ satisfy $w(ge) = w(e)$.

(c) Let $g \in G$. Assume that the set E is finite. We say that g is *E-even* if the action of g on E (that is, the permutation of E that sends every $e \in E$ to ge) is an even permutation of E.

(d) If X is any set, then the set X^E of all maps $E \to X$ becomes a G-set as follows: For any $\pi \in X^E$ and $g \in G$, we let $g\pi \in X^E$ be the map sending each $e \in E$ to $\pi(g^{-1}e)$.

(e) Let F be a further G-set. Assume that the set E is finite. An element $\pi \in F$ is said to be *E-coeven* if every $g \in G$ satisfying $g\pi = \pi$ is E-even. A G-orbit O on F is said to be *E-coeven* if all elements of O are E-coeven.[6]

Theorem 10. *Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let $\operatorname{Par}\mathbf{E}$ denote the set of all $\mathbf{E}$-partitions. Let $w : E \to \{1, 2, 3, \ldots\}$. Let G be a finite group which acts on E. Assume that G preserves both relations $<_1$ and $<_2$, and also preserves w. Then, G acts also on the set $\operatorname{Par}\mathbf{E}$ of all $\mathbf{E}$-partitions; namely, $\operatorname{Par}\mathbf{E}$ is a G-subset of the G-set $\{1, 2, 3, \ldots\}^E$ (see Definition 9 (d) for the definition of the latter). For any G-orbit O on $\operatorname{Par}\mathbf{E}$, we define a monomial $\mathbf{x}_{O,w}$ by*

$$\mathbf{x}_{O,w} = \mathbf{x}_{\pi,w} \qquad \textit{for some element } \pi \textit{ of } O$$

(this does not depend on the choice of π). Let

$$\Gamma(\mathbf{E}, w, G) = \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \mathbf{x}_{O,w}$$

and

$$\Gamma^+(\mathbf{E}, w, G) = \sum_{O \text{ is an } E\text{-coeven } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \mathbf{x}_{O,w}.$$

Then, $\Gamma(\mathbf{E}, w, G)$ and $\Gamma^+(\mathbf{E}, w, G)$ belong to QSym *and satisfy*

$$S(\Gamma(\mathbf{E}, w, G)) = (-1)^{|E|} \Gamma^+((E, >_1, <_2), w, G).$$

This theorem, which combines Theorem 7 with the ideas of Pólya enumeration, is inspired by Jochemko's reciprocity result for order polynomials [10, Theorem 2.8], which can be obtained from it by specializations (see [8, §8] for the derivation).

We shall now review a number of particular cases of Theorem 7.

Example 11. (a) Corollary 8 follows from Theorem 7 by letting w be the function which is constantly 1.

[6] Equivalently, O is E-coeven if and only if at least one element of O is E-coeven. (This is easy to check.)

(b) Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition, let $n = |\alpha|$, and let $\mathbf{E} = (E, <_1, <_2)$ be the double poset defined in Example 6 (b). Let $w : \{1, 2, \ldots, \ell\} \to \{1, 2, 3, \ldots\}$ be the map sending every i to α_i. As Example 6 (b) shows, we have $\Gamma(\mathbf{E}, w) = M_\alpha$. Thus, applying Theorem 7 to these $\mathbf{E}$ and w yields

$$S(M_\alpha) = (-1)^\ell \Gamma((E, >_1, <_2), w) = (-1)^\ell \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$
$$= (-1)^\ell \sum_{i_1 \leq i_2 \leq \cdots \leq i_\ell} x_{i_1}^{\alpha_\ell} x_{i_2}^{\alpha_{\ell-1}} \cdots x_{i_\ell}^{\alpha_1} = (-1)^\ell \sum_{\substack{\gamma \text{ is a composition; } |\gamma|=n; \\ D(\gamma) \subseteq D((\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1))}} M_\gamma.$$

This is the formula for $S(M_\alpha)$ given in [3, Proposition 3.4], in [11, (4.26)], in [9, Theorem 5.11], and in [1, Theorem 4.1] (originally due to Ehrenborg and to Malvenuto and Reutenauer).

(c) Applying Corollary 8 to the double poset of Example 6 (c) (where the relation $<_2$ is chosen to be a total order) yields a classical formula for the antipode of a fundamental quasisymmetric function ([11, (4.27)], [9, (5.9)], [1, Theorem 5.1]).

(d) By applying Corollary 8 to any of the two tertispecial double posets $\mathbf{Y}(\lambda/\mu)$ and $\mathbf{Y}_h(\lambda/\mu)$ from Example 3, we can obtain the well-known formula $S(s_{\lambda/\mu}) = (-1)^{|\lambda/\mu|} s_{\lambda^t/\mu^t}$ for the antipode of a skew Schur function (where ν^t denotes the conjugate of a partition ν). See, e.g., [8, Example 4.8 (d)] for the details. (This is not a new argument; it appeared, e.g., in [9, proof of Corollary 5.29] in the language of P-partitions. It makes use of the fact that the antipode of the symmetric functions is a restriction of the antipode of QSym.) A more general antipode formula for "Schur functions with cell weights" (no longer symmetric, at least in general) can be obtained using Theorem 7.

(e) A result of Benedetti and Sagan [1, Theorem 8.2] on the antipodes of immaculate functions can be obtained from Corollary 8 using dualization.

5 An outline of the proofs

In preparation for the proofs of the above results, we shall now introduce the notion of a *packed map*, and state some simple lemmas. Proofs can be found in [8, §5].

Definition 12. If E is a set and $\pi : E \to \{1, 2, 3, \ldots\}$ is a map, then π is said to be *packed* if $\pi(E) = \{1, 2, \ldots, k\}$ for some $k \in \mathbb{N}$.

Definition 13. Let E be a set. Let $\pi : E \to \{1, 2, 3, \ldots\}$ be a packed map. Let $w : E \to \{1, 2, 3, \ldots\}$ be a map. Then, a composition $\operatorname{ev}_w \pi$ is defined as follows: Let $\ell = |\pi(E)|$. Set $\operatorname{ev}_w \pi = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, where each $\alpha_i = \sum_{e \in \pi^{-1}(i)} w(e)$.

Proposition 14. *Let* $\mathbf{E} = (E, <_1, <_2)$ *be a double poset. Let* $w : E \to \{1, 2, 3, \ldots\}$ *be a map. Then,*

$$\Gamma(\mathbf{E}, w) = \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} M_{\operatorname{ev}_w \varphi}. \tag{5.1}$$

We shall now describe the coproduct of $\Gamma(\mathbf{E}, w)$, following [14, Theorem 2.2].

Definition 15. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset.

(a) Then, $\operatorname{Adm} \mathbf{E}$ will mean the set of all pairs (P, Q), where P and Q are subsets of E satisfying $P \cap Q = \varnothing$ and $P \cup Q = E$ and having the property that no $p \in P$ and $q \in Q$ satisfy $q <_1 p$. These pairs (P, Q) are called the *admissible partitions* of $\mathbf{E}$.

(b) For any subset T of E, we let $\mathbf{E} \mid_T$ denote the double poset $(T, <_1, <_2)$, where $<_1$ and $<_2$ (by abuse of notation) denote the restrictions of the relations $<_1$ and $<_2$ to T.

Proposition 16. *Let* $\mathbf{E} = (E, <_1, <_2)$ *be a double poset. Let* $w : E \to \{1, 2, 3, \ldots\}$ *be a map. Then,*

$$\Delta(\Gamma(\mathbf{E}, w)) = \sum_{(P,Q) \in \operatorname{Adm} \mathbf{E}} \Gamma(\mathbf{E} \mid_P, w \mid_P) \otimes \Gamma\left(\mathbf{E} \mid_Q, w \mid_Q\right). \tag{5.2}$$

Proof outline for Theorem 7. We shall only demonstrate the cornerstones of this proof. See [8, §6] for the details.

We use strong induction over $|E|$. The induction base ($|E| = 0$) is straightforward. Now, consider a tertispecial double poset $\mathbf{E} = (E, <_1, <_2)$ with $|E| > 0$ and a map $w : E \to \{1, 2, 3, \ldots\}$, and assume that Theorem 7 is proven for all tertispecial double posets of smaller size.

From $|E| > 0$, it is easy to see that $\varepsilon(\Gamma(\mathbf{E}, w)) = 0$, so that $(u \circ \varepsilon)(\Gamma(\mathbf{E}, w)) = 0$. But (2.1) yields $(m \circ (S \otimes \operatorname{id}) \circ \Delta)(\Gamma(\mathbf{E}, w)) = (u \circ \varepsilon)(\Gamma(\mathbf{E}, w)) = 0$, so that

$$\begin{aligned} 0 &= (m \circ (S \otimes \operatorname{id}) \circ \Delta)(\Gamma(\mathbf{E}, w)) = m((S \otimes \operatorname{id})(\Delta(\Gamma(\mathbf{E}, w)))) \\ &= m\left((S \otimes \operatorname{id})\left(\sum_{(P,Q) \in \operatorname{Adm} \mathbf{E}} \Gamma(\mathbf{E} \mid_P, w \mid_P) \otimes \Gamma\left(\mathbf{E} \mid_Q, w \mid_Q\right)\right)\right) \qquad \text{(by (5.2))} \\ &= \sum_{(P,Q) \in \operatorname{Adm} \mathbf{E}} S(\Gamma(\mathbf{E} \mid_P, w \mid_P))\, \Gamma\left(\mathbf{E} \mid_Q, w \mid_Q\right). \end{aligned} \tag{5.3}$$

In order to prove Theorem 7, it now suffices to verify

$$0 = \sum_{(P,Q) \in \operatorname{Adm} \mathbf{E}} (-1)^{|P|}\, \Gamma((P, >_1, <_2), w \mid_P)\, \Gamma\left(\mathbf{E} \mid_Q, w \mid_Q\right). \tag{5.4}$$

Indeed, each addend on the right hand side of (5.3) equals the corresponding addend on the right hand side of (5.4) except maybe the addend for $(P, Q) = (E, \varnothing)$ (see footnote[7]). Therefore, once (5.4) is proven, it will follow that the addends for $(P, Q) = (E, \varnothing)$ are also equal; but this is precisely the claim $S(\Gamma(\mathbf{E}, w)) = (-1)^{|E|} \Gamma((E, >_1, <_2), w)$ that needs to be proven. Hence, proving (5.4) suffices.

Using the definitions of $\Gamma((P, >_1, <_2), w\mid_P)$ and $\Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right)$, we observe that each $(P, Q) \in \operatorname{Adm} \mathbf{E}$ satisfies

$$\begin{aligned}
&\Gamma((P, >_1, <_2), w\mid_P)\, \Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right) \\
&= \left(\sum_{\sigma \text{ is a } (P, >_1, <_2)\text{-partition}} \mathbf{x}_{\sigma, w|_P} \right) \left(\sum_{\tau \text{ is a } (Q, <_1, <_2)\text{-partition}} \mathbf{x}_{\tau, w|_Q} \right) \\
&= \sum_{\substack{\pi : E \to \{1,2,3,\ldots\}; \\ \pi|_P \text{ is a } (P, >_1, <_2)\text{-partition}; \\ \pi|_Q \text{ is a } (Q, <_1, <_2)\text{-partition}}} \mathbf{x}_{\pi, w}.
\end{aligned}$$

Therefore, in order to prove (5.4), it will be enough to show that for every map $\pi : E \to \{1, 2, 3, \ldots\}$, we have

$$\sum_{\substack{(P,Q) \in \operatorname{Adm} \mathbf{E}; \\ \pi|_P \text{ is a } (P, >_1, <_2)\text{-partition}; \\ \pi|_Q \text{ is a } (Q, <_1, <_2)\text{-partition}}} (-1)^{|P|} = 0. \tag{5.5}$$

Hence, let us fix a map $\pi : E \to \{1, 2, 3, \ldots\}$. Our goal is now to prove (5.5). We denote by Z the set of all $(P, Q) \in \operatorname{Adm} \mathbf{E}$ such that $\pi\mid_P$ is a $(P, >_1, <_2)$-partition and $\pi\mid_Q$ is a $(Q, <_1, <_2)$-partition. We are going to define an involution $T : Z \to Z$ of the set Z having the property that, for any $(P, Q) \in Z$, if we write $T((P, Q))$ in the form (P', Q'), then $(-1)^{|P'|} = -(-1)^{|P|}$. Once such an involution T is found, it will clearly partition the addends on the left hand side of (5.5) into pairs of mutually cancelling addends, and so (5.5) will follow and we will be done. It thus remains to find T.

The definition of T is simple: Let F be the subset of E consisting of those $e \in E$ which have minimum $\pi(e)$. Then, F is a nonempty subposet of the poset $(E, <_2)$, and hence has a minimal element f (that is, an element f such that no $g \in F$ satisfies $g <_2 f$). Fix such an f. Now, the map T sends a $(P, Q) \in Z$ to $\begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases}$.

Less simple is the proof that T is well-defined. See [8, §6] for this argument. □

We shall be particularly brief about the proof of Theorem 10; the full proof can be found in [8, §7]. We merely state the two main observations used in the proof:

[7] Because if $(P, Q) \neq (E, \varnothing)$, then $|P| < |E|$, and thus the induction hypothesis (applied to the double poset $\mathbf{E}\mid_P$, which is easily seen to be tertispecial) yields $S(\Gamma(\mathbf{E}\mid_P, w\mid_P)) = (-1)^{|P|} \Gamma((P, >_1, <_2), w\mid_P)$.

Proposition 17. *Let* $\mathbf{E} = (E, <_1, <_2)$ *be a tertispecial double poset. Let* G *be a finite group which acts on* E. *Assume that* G *preserves both relations* $<_1$ *and* $<_2$.

Let $g \in G$. *Let* E^g *be the set of all orbits under the action of* g *on* E. *Define a binary relation* $<_1^g$ *on* E^g *by*

$$\left(u <_1^g v\right) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b) .$$

Define a binary relation $<_2^g$ *similarly. Set* $\mathbf{E}^g = \left(E^g, <_1^g, <_2^g\right)$.

(a) Then, $\mathbf{E}^g$ *is a tertispecial double poset.*

There is a bijection $\Phi : \{\pi : E \to \{1,2,3,\ldots\} \mid g\pi = \pi\} \to \{\overline{\pi} : E^g \to \{1,2,3,\ldots\}\}$. *Namely, this bijection* Φ *sends any map* $\pi : E \to \{1,2,3,\ldots\}$ *satisfying* $g\pi = \pi$ *to the map* $\overline{\pi} : E^g \to \{1,2,3,\ldots\}$ *defined by*

$$\overline{\pi}\,(u) = \pi\,(a) \qquad \text{for every } u \in E^g \text{ and } a \in u.$$

Consider this bijection Φ. *Let* $\pi : E \to \{1,2,3,\ldots\}$ *be a map satisfying* $g\pi = \pi$.

(b) The map π *is an* $\mathbf{E}$*-partition if and only if the map* $\Phi\,(\pi)$ *is an* $\mathbf{E}^g$*-partition.*

(c) Let $w : E \to \{1,2,3,\ldots\}$ *be map. Define a map* $w^g : E^g \to \{1,2,3,\ldots\}$ *by*

$$w^g\,(u) = \sum_{a \in u} w\,(a) \qquad\qquad \text{for every } u \in E^g.$$

Then, $\mathbf{x}_{\Phi(\pi),w^g} = \mathbf{x}_{\pi,w}$.

Lemma 18. *Let* G *be a finite group. Let* F *be a* G*-set. Let* O *be a* G*-orbit on* F, *and let* $\pi \in O$.

(a) We have $\dfrac{1}{|O|} = \dfrac{1}{|G|} \sum\limits_{\substack{g \in G; \\ g\pi = \pi}} 1$.

(b) Let E *be a further finite* G*-set. For every* $g \in G$, *let* $\operatorname{sign}_E g$ *denote the sign of the permutation of* E *that sends every* $e \in E$ *to* ge. *(Thus,* $g \in G$ *is* E*-even if and only if* $\operatorname{sign}_E g = 1$.*) Then,* $\begin{cases} \dfrac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} = \dfrac{1}{|G|} \sum\limits_{\substack{g \in G; \\ g\pi = \pi}} \operatorname{sign}_E g.$

Theorem 10 can be derived from Theorem 7 using the above observations and some standard manipulations of sums, akin to the proof of the Pólya enumeration formula.

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